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Reflection and Transmission of Electromagnetic Waves by a Spherical Shell

by

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and

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REFLECTION AND TRANSMISSION OF ELECTROMAGNETIC WAVES BY A SPHERICAL SHELL

by

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ABSTRACT

Exact, explicit expressions are obtained for the field of a radiating periodic point dipole located in medium 1 at the center of a spherical shell of medium 2, which is bounded by medium 3. The values of ε, μ, σ and the shell radii are arbitrary. The reflected and transmitted fields are examined in various special cases. When the shell radii become infinite, the fields agree with those obtained by R. K. Luneberg for the case of plane waves normally incident on a flat plate of medium 2 between media 1 and 3. When the radii are infinite and media 2 and 3 identical, the Fresnel formulae for normal incidence of plane waves on a plane interface between two half-infinite media are obtained. When the radii are finite, media 2 and 3 identical and $\mu_1 = \mu_3$, the results of C. T. Tai are obtained. For finite radii, media 1 and 3 identical, and shell thickness small compared to a wavelength, the fields check the approximate results of J. B. Keller for the fields reflected and transmitted by a thin shell of any shape, when his results are specialized to the present case.

Introduction

The electromagnetic field due to a radiating periodic point dipole in a homogeneous isotropic medium of infinite extent is well known[1]. In this paper we intend to find the modification of the field when the source is at the center of a medium bounded by a spherical shell of different material, with a third medium of infinite extent external to the shell. It is expected that part of the field will be reflected from the shell and part transmitted. These reflected and transmitted fields are of particular interest. This problem is solved exactly and the solution is examined in special cases, in some of which it reduces to known results. In addition, the solution is used to test the accuracy of a recent approximate calculation of the reflection and transmission by an arbitrarily curved thin shell[2].

Formulation of the Problem

A point dipole with axis in the z-direction is at the origin of coordinates in medium 1, with constants μ_1, ε_1, and σ_1, and bounded at $r = r_a$ by medium 2. This medium extends from $r = r_a$ to $r = r_b$ and has the constants μ_2, ε_2, and σ_2. Bounding the shell at $r = r_b$ is medium 3 of infinite extent and with constants μ_3, ε_3, and σ_3.

The electromagnetic field vectors E and H must satisfy Maxwell's equations and appropriate boundary conditions at the two interfaces between the shell and the surrounding media. In addition the field must vanish at infinity and also contain the incident field. We assume the time dependence of the field components to be of the form $e^{i\omega t}$, where ω is the angular frequency of the dipole. With this time factor, no volume charge present, and μ, ε, and σ each constant, Maxwell's equa-

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1. Stratton, J. A.; Electromagnetic Theory, McGraw-Hill, 1941, New York, pp 436-7.
   Ramo, S. and J. R. Whinnery, Fields and Waves in Modern Radio, Wiley, 1944, New York, p 430.
2. Keller, Joseph B., "Reflection and Transmission of Electromagnetic Waves by Thin Curved Shells", Washington Square College Mathematics Research Group, Research Report No. 172-5, Jan.1948, Report prepared and written under Contract No. W28-099-ac-172 with Army Air Forces.

tions become[3]:

$$\nabla \cdot E = 0$$
$$\nabla \cdot H = 0$$
$$\nabla \times E = -\frac{i\omega\mu}{c} H \qquad (1)$$
$$\nabla \times H = \frac{i\omega\epsilon'}{c} E$$

where

$$\epsilon' = \epsilon - \frac{4\pi i\sigma}{\omega} \qquad (2)$$

and the constants have the appropriate values in each medium.

The boundary conditions, requiring continuity of the tangential components of E and H and the proper jump-discontinuity of the normal components of the fields, are given by[4]:

at $r = r_a$	at $r = r_b$
$E_{1_t} = E_{2_t}$	$E_{2_t} = E_{3_t}$
$H_{1_t} = H_{2_t}$	$H_{2_t} = H_{3_t}$
$\epsilon'_1 E_{1_n} = \epsilon'_2 E_{2_n}$	$\epsilon'_2 E_{2_n} = \epsilon'_3 E_{3_n}$
$\mu_1 H_{1_n} = \mu_2 H_{2_n}$	$\mu_2 H_{2_n} = \mu_3 H_{3_n}$

(3)

In these equations E_{1_t} represents the tangential component of E in medium 1, E_{2_t} the tangential component of E in medium 2, E_{1_n} the normal component of E in medium 1, etc.

Use of Hertz Vector

We introduce the Hertz vector A which satisfies

$$\nabla^2 A + \frac{\omega^2\mu\epsilon'}{c^2} A = 0 \qquad (4)$$

It is easily verified that the field vectors will satisfy Maxwell's equations if we define them in terms of the Hertz vector as[5]

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3. Ramo, S. and J. R. Whinnery, op. cit. p 154.
4. Stratton, J. A., op. cit. pp 483-4.
5. Ramo, S. and J. R. Whinnery, op. cit. p 165.

$$H = \nabla \times A$$

$$E = \frac{c}{i\omega\epsilon'} \nabla(\nabla \cdot A) - \frac{i\omega\mu}{c} A \qquad (5)$$

For a point dipole with axis in the z-direction, in a medium of infinite extent, the solution of equation (4) is found to be a vector with a single component[6]

$$A_z = A_o \frac{e^{-ikr}}{r} \qquad (6)$$

where A_o is a constant proportional to the dipole strength, and

$$k = \frac{\omega}{c}\sqrt{\mu\epsilon'} \qquad (7)$$

Solution of the Problem

In the present case we assume the solution of equation (4) to be a vector with the single non-zero component:

$$\begin{aligned} A_z &= A_o \frac{e^{-ik_1 r}}{r} + A_1 \frac{e^{ik_1 r}}{r} && 0 < r < r_a \\ A_z &= A_2 \frac{e^{-ik_2 r}}{r} + A_3 \frac{e^{ik_2 r}}{r} && r_a < r < r_b \\ A_z &= A_4 \frac{e^{-ik_3 r}}{r} && r_b < r \end{aligned} \qquad (8)$$

In this solution the constant A_o is presumed to be known since it is determined by the dipole strength, and the constants A_1, A_2, A_3, A_4 are to be determined. From equations (8) it can be seen that the assumed solution contains the incident field (6) and vanishes at infinity. By substitution one finds that (8) is actually a solution of equation (4), and therefore the field components, which are obtained by using (8) in equations (5), will satisfy Maxwell's equations in each region. These

---

6. ibid. p 430

field components are:

$$H_{\varphi_1} = \left\{ A_o e^{-ik_1 r}\left[ik_1 + \frac{1}{r}\right] + A_1 e^{ik_1 r}\left[-ik_1 + \frac{1}{r}\right]\right\} \frac{1}{r} \sin\theta$$

$$H_{\varphi_2} = \left\{ A_2 e^{-ik_2 r}\left[ik_2 + \frac{1}{r}\right] + A_3 e^{ik_2 r}\left[-ik_2 + \frac{1}{r}\right]\right\} \frac{1}{r} \sin\theta \qquad (9)$$

$$H_{\varphi_3} = \left\{ A_4 e^{-ik_3 r}\left[ik_3 + \frac{1}{r}\right]\right\}\frac{1}{r} \sin\theta$$

$$E_{r_1} = \left\{ A_o e^{-ik_1 r}\left[ik_1 + \frac{1}{r}\right] + A_1 e^{ik_1 r}\left[-ik_1 + \frac{1}{r}\right]\right\} \frac{2c}{i\omega \varepsilon'_1 r^2} \cos\theta$$

$$E_{r_2} = \left\{ A_2 e^{-ik_2 r}\left[ik_2 + \frac{1}{r}\right] + A_3 e^{ik_2 r}\left[-ik_2 + \frac{1}{r}\right]\right\} \frac{2c}{i\omega \varepsilon'_2 r^2} \cos\theta \qquad (10)$$

$$E_{r_3} = \left\{ A_4 e^{-ik_3 r}\left[ik_3 + \frac{1}{r}\right]\right\}\frac{2c}{i\omega \varepsilon'_3 r^2} \cos\theta$$

$$E_{\theta_1} = \left\{ A_o e^{-ik_1 r}\left[ik_1^2 + \frac{k_1}{r} - \frac{i}{r^2}\right] + A_1 e^{ik_1 r}\left[ik_1^2 - \frac{k_1}{r} - \frac{i}{r^2}\right]\right\}\frac{c}{\omega \varepsilon'_1 r} \sin\theta$$

$$E_{\theta_2} = \left\{ A_2 e^{-ik_2 r}\left[ik_2^2 + \frac{k_2}{r} - \frac{i}{r^2}\right] + A_3 e^{ik_2 r}\left[ik_2^2 - \frac{k_2}{r} - \frac{i}{r^2}\right]\right\}\frac{c}{\omega \varepsilon'_2 r} \sin\theta \qquad (11)$$

$$E_{\theta_3} = \left\{ A_4 e^{-ik_3 r}\left[ik_3^2 + \frac{k_3}{r} - \frac{i}{r^2}\right]\right\} \frac{c}{\omega \varepsilon'_3 r} \sin\theta$$

$$H_{\theta_i} = H_{r_i} = E_{\varphi_i} = 0, \qquad (i = 1,2,3) \qquad (12)$$

From equations (9) and (10) we see that, in each region, H_φ and E_r are related by

$$H_\varphi = \varepsilon' E_r \left(\frac{i\omega r}{2c}\right) \tan\theta \qquad (13)$$

We must now determine A_i (i=1,2,3,4) so that the field components (9)-(12) will satisfy the appropriate boundary conditions (3)at the two interfaces. On account of equations (12) we have $E_t = E_\theta$, $H_t = H_\varphi$, and of course $E_n = E_r$ and $H_n = H_r$. From (12) it immediately follows that the condition on the normal component of H is satisfied identically at both boundaries. If H_φ satisfies the condition presented by equations (3) at the interfaces then, by virtue of equation (13), E_r will satisfy the condition for E_n as given in (3). Thus the field components (9)-(12) will satisfy Maxwell's equations and the boundary conditions provided A_i (i=1,2,3,4) are determined by:

$$
\begin{aligned}
B_{oa} A_o &= -B_{1a}A_1 + B_{2a} A_2 + B_{3a} A_3 && ; \text{ at } r=r_a,\ H_{\varphi_1} = H_{\varphi_2} \\
0 &= B_{2b} A_2 + B_{3b}A_3 - B_{4b}A_4 && , \text{ at } r=r_b,\ H_{\varphi_2} = H_{\varphi_3} \\
C_{oa} A_o &= -C_{1a}A_1 + C_{2a} A_2 + C_{3a} A_3 && ; \text{ at } r=r_a,\ E_{\theta_1} = E_{\theta_2} \\
0 &= C_{2b} A_2 + C_{3b} A_3 - C_{4b}A_4 && ; \text{ at } r=r_b,\ E_{\theta_2} = E_{\theta_3}
\end{aligned}
\qquad (14)
$$

where the following abbreviations have been used:

$$
\begin{aligned}
B_{oa} &= e^{-ik_1 r_a}\left[ik_1 + \frac{1}{r_a} \right] & C_{oa} &= \frac{e^{-ik_1 r_a}}{\varepsilon_1'}\left[ik_1^2 + \frac{k_1}{r_a} - \frac{i}{r_a^2} \right] \\
B_{1a} &= e^{ik_1 r_a}\left[-ik_1 + \frac{1}{r_a} \right] & C_{1a} &= \frac{e^{ik_1 r_a}}{\varepsilon_1'}\left[ik_1^2 - \frac{k_1}{r_a} - \frac{i}{r_a^2} \right] \\
B_{2a} &= e^{-ik_2 r_a}\left[ik_2 + \frac{1}{r_a} \right] & C_{2a} &= \frac{e^{-ik_2 r_a}}{\varepsilon_2'}\left[ik_2^2 + \frac{k_2}{r_a} - \frac{i}{r_a^2} \right] \\
B_{3a} &= e^{ik_2 r_a}\left[-ik_2 + \frac{1}{r_a} \right] & C_{3a} &= \frac{e^{ik_2 r_a}}{\varepsilon_2'}\left[ik_2^2 - \frac{k_2}{r_a} - \frac{i}{r_a^2} \right] \\
B_{2b} &= e^{-ik_2 r_b}\left[ik_2 + \frac{1}{r_b} \right] & C_{2b} &= \frac{e^{-ik_2 r_b}}{\varepsilon_2'}\left[ik_2^2 + \frac{k_2}{r_b} - \frac{i}{r_b^2} \right] \\
B_{3b} &= e^{ik_2 r_b}\left[-ik_2 + \frac{1}{r_b} \right] & C_{3b} &= \frac{e^{ik_2 r_b}}{\varepsilon_2'}\left[ik_2^2 - \frac{k_2}{r_b} - \frac{i}{r_b^2} \right]
\end{aligned}
\qquad (15)
$$

$$B_{4b} = e^{-ik_3 r_b}\left[ik_3 + \frac{1}{r_b}\right] \qquad C_{4b} = \frac{e^{-ik_3 r_b}}{\varepsilon_3'}\left[ik_3^2 + \frac{k_3}{r_b} - \frac{i}{r_b^2}\right] \tag{15}$$

The four linear equations (14) in the four unknowns A_1, A_2, A_3, A_4 have the solution:

$$\begin{aligned}
A_1 &= -\frac{A_o}{D}\left\{(B_{4b}C_{3b} - B_{3b}C_{4b})(B_{oa}C_{2a} - B_{2a}C_{oa}) + (B_{4b}C_{2b} - B_{2b}C_{4b})(B_{3a}C_{oa} - B_{oa}C_{3a})\right\} \\
A_2 &= \frac{A_o}{D}\left\{(B_{4b}C_{3b} - B_{3b}C_{4b})(B_{1a}C_{oa} - B_{oa}C_{1a})\right\} \\
A_3 &= \frac{A_o}{D}\left\{(B_{4b}C_{2b} - B_{2b}C_{4b})(B_{oa}C_{1a} - B_{1a}C_{oa})\right\} \\
A_4 &= \frac{A_o}{D}\left\{(B_{3b}C_{2b} - B_{2b}C_{3b})(B_{oa}C_{1a} - B_{1a}C_{oa})\right\}
\end{aligned} \tag{16}$$

where

$$D = \left\{(B_{4b}C_{3b} - B_{3b}C_{4b})(B_{1a}C_{2a} - B_{2a}C_{1a}) + (B_{4b}C_{2b} - B_{2b}C_{4b})(B_{3a}C_{1a} - B_{1a}C_{3a})\right\}$$

The values of A_i can be found explicitly by inserting in (16) the expressions given in (15). Thus an exact solution for the Hertz vector is given by putting into equations (8) the values of A_i given by (16), and the fields are given by putting these values of A_i into equations (9)-(12).

Discussion of the Solution

From equations (8), (15), and (16) we obtain the Hertz vectors of the reflected and transmitted fields. They are

$$A_1 \frac{e^{ik_1 r}}{r} = -A_o \frac{e^{-i\left[k_1 r_a + k_1(r_a - r)\right]}}{r}\left[\frac{e^{ik_2 h}\, F \cdot M + e^{-ik_2 h}\, J \cdot P}{e^{ik_2 h}\, F \cdot G + e^{-ik_2 h}\, J \cdot L}\right] \tag{17}$$

$$A_4 \frac{e^{-ik_3 r}}{r} = A_o\, \varepsilon_2'\, \varepsilon_3' \frac{e^{-i\left[k_1 r_a + k_3(r - r_b)\right]}}{r}\left[\frac{Q \cdot S}{e^{ik_2 h}\, F \cdot G + e^{-ik_2 h}\, J \cdot L}\right] \tag{18}$$

where the new symbols are defined by:

$$F = \left\{\varepsilon_3' \left[ik_3 + \frac{1}{r_b}\right]\left[ik_2^2 - \frac{k_2}{r_b} - \frac{i}{r_b^2}\right] - \varepsilon_2'\left[-ik_2 + \frac{1}{r_b}\right]\left[ik_3^2 + \frac{k_3}{r_b} - \frac{i}{r_b^2}\right]\right\}$$

$$G = \left\{\varepsilon_1' \left[-ik_1 + \frac{1}{r_a}\right]\left[ik_2^2 + \frac{k_2}{r_a} - \frac{i}{r_a^2}\right] - \varepsilon_2' \left[ik_2 + \frac{1}{r_a}\right]\left[ik_1^2 - \frac{k_1}{r_a} - \frac{i}{r_a^2}\right]\right\}$$

$$J = \left\{\varepsilon_3' \left[ik_3 + \frac{1}{r_b}\right]\left[ik_2^2 + \frac{k_2}{r_b} - \frac{i}{r_b^2}\right] - \varepsilon_2' \left[ik_2 + \frac{1}{r_b}\right]\left[ik_3^2 + \frac{k_3}{r_b} - \frac{i}{r_b^2}\right]\right\}$$

$$L = \left\{\varepsilon_2' \left[-ik_2 + \frac{1}{r_a}\right]\left[ik_1^2 - \frac{k_1}{r_a} - \frac{i}{r_a^2}\right] - \varepsilon_1' \left[-ik_1 + \frac{1}{r_a}\right]\left[ik_2^2 - \frac{k_2}{r_a} - \frac{i}{r_a^2}\right]\right\}$$

$$M = \left\{\varepsilon_1' \left[ik_1 + \frac{1}{r_a}\right]\left[ik_2^2 + \frac{k_2}{r_a} - \frac{i}{r_a^2}\right] - \varepsilon_2' \left[ik_2 + \frac{1}{r_a}\right]\left[ik_1^2 + \frac{k_1}{r_a} - \frac{i}{r_a^2}\right]\right\} \quad (19)$$

$$P = \left\{\varepsilon_2' \left[-ik_2 + \frac{1}{r_a}\right]\left[ik_1^2 + \frac{k_1}{r_a} - \frac{i}{r_a^2}\right] - \varepsilon_1' \left[ik_1 + \frac{1}{r_a}\right]\left[ik_2^2 - \frac{k_2}{r_a} - \frac{i}{r_a^2}\right]\right\}$$

$$Q = \left\{\left[-ik_2 + \frac{1}{r_b}\right]\left[ik_2^2 + \frac{k_2}{r_b} - \frac{i}{r_b^2}\right] - \left[ik_2 + \frac{1}{r_b}\right]\left[ik_2^2 - \frac{k_2}{r_b} - \frac{i}{r_b^2}\right]\right\}$$

$$S = \left\{\left[ik_1 + \frac{1}{r_a}\right]\left[ik_1^2 - \frac{k_1}{r_a} - \frac{i}{r_a^2}\right] - \left[-ik_1 + \frac{1}{r_a}\right]\left[ik_1^2 + \frac{k_1}{r_a} - \frac{i}{r_a^2}\right]\right\}$$

$$h = r_b - r_a$$

The Hertz vectors of the reflected and transmitted fields given by equations (17) and (18), assume simpler forms in special cases, some of which will now be examined.

I. When h=0 or $r_a = r_b$ the shell of medium 2 is absent. In this case equations (17) and (18) become:

$$A_1 \frac{e^{ik_1 r}}{r} = A_o \frac{e^{-i\left[k_1 r_a + k_1(r_a - r)\right]}}{r}$$

$$\left\{ \frac{\left[k_1 k_3(\varepsilon'_3 k_1 - \varepsilon'_1 k_3) - \frac{(k_1+k_3)}{r_a^2}(\varepsilon'_3 - \varepsilon'_1)\right] - i\left[\frac{(k_1+k_3)}{r_a}(\varepsilon'_3 k_1 - \varepsilon'_1 k_3) - \frac{(\varepsilon'_3 - \varepsilon'_1)}{r_a^3}\right]}{\left[-k_1 k_3(\varepsilon'_3 k_1 + \varepsilon'_1 k_3) + \frac{(k_3-k_1)}{r_a^2}(\varepsilon'_3 - \varepsilon'_1)\right] - i\left[\frac{(k_3-k_1)}{r_a}(\varepsilon'_3 k_1 + \varepsilon'_1 k_3) - \frac{(\varepsilon'_3 - \varepsilon'_1)}{r_a^3}\right]} \right\} \tag{20}$$

$$A_4 \frac{e^{-ik_3 r}}{r} = A_o \frac{e^{-i\left[k_1 r_a + k_3(r - r_a)\right]}}{r}$$

$$\left\{ \frac{-2\varepsilon'_3 k_1^3}{\left[-k_1 k_3(\varepsilon'_3 k_1 + \varepsilon'_1 k_3) + \frac{(k_3-k_1)}{r_a^2}(\varepsilon'_3 - \varepsilon'_1)\right] - i\left[\frac{(k_3-k_1)}{r_a}(\varepsilon'_3 k_1 + \varepsilon'_1 k_3) - \frac{(\varepsilon'_3 - \varepsilon'_1)}{r_a^3}\right]} \right\} \tag{21}$$

If, in addition, r_a becomes infinite while $r - r_a$ remains finite and $A_o \frac{e^{-ik_1 r_a}}{r}$ approaches a limit A'_o, the problem reduces to that of a plane wave normally incident on a plane interface between two half-infinite media. In this case, writing $r - r_a = x$, equations (20) and (21) reduce to:

$$A_1 \frac{e^{+ik_1 r}}{r} = A'_o e^{ik_1 x} \left\{ \frac{\varepsilon'_1 k_3 - \varepsilon'_3 k_1}{\varepsilon'_1 k_3 + \varepsilon'_3 k_1} \right\} \tag{22}$$

$$A_4 \frac{e^{-ik_3 r}}{r} = A'_o e^{-ik_3 x} \left\{ \frac{2\varepsilon'_3 \frac{k_1^2}{k_3}}{\varepsilon'_1 k_3 + \varepsilon'_3 k_1} \right\} \tag{23}$$

Equations (22) and (23) are the well-known Fresnel formulae for normal incidence[7] and equations (20) and (21) are generalizations of these formulae. If $\mu_1 = \mu_3$ equations (20) and (21) reduce to those obtained by C.T.Tai[8] for this special case.

II. When r_b and r_a become infinite while $r_b - r_a = h$ and $r - r_a = x$ remain finite and $A_o \frac{e^{-ik_1 r_a}}{r}$ approaches a limit A_o', the problem reduces to that of plane waves normally incident on a flat plate separating two different half-infinite media. In this case equations (17) and (18) become:

$$A_1 \frac{e^{ik_1 r}}{r} = A_o' e^{ik_1 x} \cdot$$

$$\left\{ \frac{\varepsilon_2' k_2 (\varepsilon_1' k_3 - \varepsilon_3' k_1) \cos k_2 h + i (\varepsilon_1' \varepsilon_3' k_2^2 - \varepsilon_2'^2 k_1 k_3) \sin k_2 h}{\varepsilon_2' k_2 (\varepsilon_1' k_3 + \varepsilon_3' k_1) \cos k_2 h + i (\varepsilon_1' \varepsilon_3' k_2^2 + \varepsilon_2'^2 k_1 k_3) \sin k_2 h} \right\} \qquad (24)$$

$$A_4 \frac{e^{-ik_3 r}}{r} = A_o' e^{-ik_3 (x-h)} \cdot$$

$$\left\{ \frac{2 \varepsilon_2' \varepsilon_3' k_1^2 \frac{k_2}{k_3}}{\varepsilon_2' k_2 (\varepsilon_1' k_3 + \varepsilon_3' k_1) \cos k_2 h + i (\varepsilon_1' \varepsilon_3' k_2^2 + \varepsilon_2'^2 k_1 k_3) \sin k_2 h} \right\} \qquad (25)$$

Equations (24) and (25) are the same as those obtained by R. K. Luneberg for this case[9].

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7. Stratton, J.A., op. cit. p 492

8. Tai, C.T., "Radiation of a Hertzian Dipole Immersed in a Dissipative Medium", Harvard University, Cruft Laboratory, Research Report No. 21.

9. Luneberg, R.K., "The Propagation of Electromagnetic Plane Waves in Plane Parallel Layers", Washington Square College Mathematics Research Group, Research Report No. 172-3, June 1947, pp 48-9. Report prepared and written under Contract No. W28-099-ac-172 with Army Air Forces.

III. If $h/r_a \ll 1$ and $|k_2 h| \ll 1$ then the Hertz vectors given by equations (17) and (18) are accurately represented by the first few terms in their Taylor's expansions in powers of h. The first terms of these expansions are given below for the case in which medium 3 is the same as medium 1, and $|1/k_1 r_a| \ll 1$.

$$A_1 \frac{e^{ik_1 r}}{r} = A_o \frac{e^{-i[2k_1 r_a - k_1 r]}}{r} \frac{ik_2 h}{2} \left\{ \frac{\varepsilon_1' k_2}{\varepsilon_2' k_1} - \frac{\varepsilon_2' k_1}{\varepsilon_1' k_2} \right\} \tag{25}$$

$$A_4 \frac{e^{-ik_1 r}}{r} = A_o \frac{e^{-ik_1 r}}{r} \left\{ 1 + ik_1 h - \frac{ik_2 h}{2} \left[\frac{\varepsilon_1' k_2}{\varepsilon_2' k_1} + \frac{\varepsilon_2' k_1}{\varepsilon_1' k_2} \right] \right\} \tag{26}$$

By using (25) and (26) in equations (9)-(12) we obtain the reflected and transmitted field components expanded in powers of h. These are, when $\frac{1}{|k_1 r|} \ll 1$,

$$H_{Q_R} = A_1 \frac{e^{ik_1 r}}{r} (-ik_1 \sin\Theta) \tag{27}$$

$$H_{Q_T} = A_4 \frac{e^{-ik_1 r}}{r} (ik_1 \sin\Theta) \tag{28}$$

$$E_{\Theta_R} = A_1 \frac{e^{ik_1 r}}{r} \left(\frac{ick_1^2}{\omega \varepsilon_1'} \sin\Theta\right) \tag{29}$$

$$E_{\Theta_T} = A_4 \frac{e^{-ik_1 r}}{r} \left(\frac{ick_1^2}{\omega \varepsilon_1'} \sin\Theta\right) \tag{30}$$

$$E_{r_R} = A_1 \frac{e^{ik_1 r}}{r} \left(- \frac{2ck_1}{\omega \varepsilon_1' r} \cos\Theta \right) \tag{31}$$

$$E_{r_T} = A_4 \frac{e^{-ik_1 r}}{r} \left(\frac{2ck_1}{\omega \varepsilon_1' r} \cos\Theta\right) \tag{32}$$

$$H_r = H_\Theta = E_Q = 0 \tag{33}$$

In these equations the letters R and T in the subscripts denote reflected and transmitted field components, respectively.

Comparison with an Approximate Solution

An approximate solution of the reflection and transmission of plane or spherical electromagnetic waves from an arbitrarily curved thin shell has recently been given by J. B. Keller[10]. The results, when specialized to the case of plane waves incident on a thin flat plate, agree with those of R. K. Luneberg[11]. In order to obtain another check on the approximate solution we will specialize it to the case of a point source at the center of a spherical shell. The fields obtained in this way should agree with those given by equations (27)-(33) above.

When equation (45) in the paper by J. B. Keller and the corresponding equation for H are specialized to this case it is found that the reflected and transmitted H_r, H_Θ, and E_Q are all zero. We also find that the reflected and transmitted H_Q and E_Θ are given by equations (27)-(30). The reflected and transmitted E_r components become:

$$E_{r_R} = A_1 \frac{e^{ik_1 r}}{r} \left(- \frac{2ck_1}{\omega \varepsilon_1' r} \cos \Theta \right) \frac{r}{r_a} \qquad (34)$$

$$E_{r_T} = \left[A_0 + (A_4 - A_0) \frac{r}{r_a} \right] \frac{e^{-ik_1 r}}{r} \left(\frac{2ck_1}{\omega \varepsilon_1' r} \cos \Theta \right) \qquad (35)$$

We thus see that the approximate solution agrees with the results of the special form of the exact solution given by equations (27)-(33) in all components except E_{r_R} and E_{r_T}. Since equations (27)-(32) are valid only when $|k_1 r_a| >> 1$ and $|k_1 r| >> 1$, the approximate solution can give good results only under the same conditions. The first condition was to be expected from the derivation of the approximate solution, which assumed that the source was many wave-lengths from the obstacle. In that determination it was also assumed that there was no singularity in the reflected field. In the special case here considered there is a singularity

10. See reference in footnote 2 on p 1.

11. See reference in footnote 9 on p 9.

at $r = 0$ and hence it is to be expected that the result will hold only at a distance of several wave-lengths from the singularity, i.e., $| k_1 r | \gg 1$.

In the case of the components E_{r_R} and E_{r_T}, equations (34) and (35) become identical with equations (31) and (32) when $r = r_a$, but differ for other values of r. The error in the radial dependence of E_r is due to the sensitive dependence of E_r on the method of evaluating an integral occurring in the approximate solution.

From the above comparison of the approximate and exact solutions it can be seen that the approximate solution is correct except for the longitudinal field component E_r, and except near the singularity of the reflected field. Since the occurrence of singularities in reflected fields is uncommon, as is also the occurrence of longitudinal components, the comparison indicates that the approximate solution should usually be acceptable.

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